Tutorial 5 ---Chan Ki Fung

BACK

Questions of today

1. It is shown in lecture 9 that

$$\Gamma(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{n+s}+\int_1^{\infty}e^t t^{s-1}dt.$$

   Show that the second term on the right extends to an entire function.
2. Show that the following limit exists:

$$\gamma=\lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{n}\right)-\log N$$

3. (Gauss formula) For $z\neq 0,-1,\dots$

$$\Gamma(z)=\lim_{z\to\infty}\frac{n!n^z}{z(z+1)\cdots(z+n)}.$$

4. (Bohr-Mollerup Theorem) Let f be a function defined on $(0,\infty)$ such that $f(x)>0$ for all $x>0$. Suppose that f has the following properties:
   1. $\log f(x)$ is a convex function;
   2. $f(x+1)=f(x)$ for all x;
   3. $f(1)=1$.

   Then $f(x)=\Gamma(x)$ for all x.
5. Show that, for $u,v>0$.

$$\Gamma(u)\Gamma(v)=2\Gamma(u+v)\int_0^{\frac{\pi}{2}}(\cos\theta)^{2u-1}(\sin\theta)^{2v-1}d\theta.$$

6. For $\mathrm{Re}(z)>2$, show that
   1. $$\zeta(z)^2=\sum_{n=1}^{\infty}\frac{d(n)}{n^z}$$

      where $d(n)$ is the number of divisors of n.
   2. $$\zeta(z)\zeta(z-1)=\sum_{n=1}^{\infty}\frac{\sigma(n)}{n^z}$$

      where $\sigma(n)$ is the sum of divisors of n.

Hints & solutions of today

1. This is homework 2.
2. Write the limit as

$$\lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{n}-\int_n^{n+1}\frac{1}{x}\right)$$

   and so the terms in the series are of $O(\frac{1}{n^2})$.
3. $$\begin{aligned}\Gamma(z)&=\frac{e^{-\gamma z}}{z}\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)^{-1}e^{z/k}\\&=\frac{e^{-\gamma z}}{z}\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{ke^{z/k}}{z+k}\right)\\&=\lim_{n\to\infty}\frac{n!n^z\exp(z(-\gamma+1+1/2+\cdots+1/n-\log n))}{z\cdot(z+1)\cdots(z+n)}\end{aligned}$$

4. Only need to show the case $0<x\leq$ Convexity implies that

$$\frac{\log f(n)-\log f(n-1)}{n-(n-1)}\leq\frac{\log f(x+n)-\log f(n)}{(x+n)-n}\leq\frac{\log f(n+1)-\log f(n)}{(n+1)-n}$$

   which you simplify and obtain

$$(n-1)^x(n-1)!\leq f(x+n)\leq n^x(n-1)!$$

   Now you property 2 to obtain

$$\frac{(n-1)^x(n-1)!}{x(x+1)\cdots(x+n-1)}\leq f(x)\leq\frac{n^x(n-1)!}{x(x+1)\cdots(x+n-1)}$$

   Finally, take $n\to\infty$ and use question 3.

$$\Gamma(u)\Gamma(v)=\int_0^{\infty}\int_0^{\infty}e^{-(x+y)}x^{u-1}y^{v-1}dxdy$$

   Then just use the substitution $x=r\cos^2\theta, y=r\sin^2\theta$.
5. We use the infinite series definition of $\zeta(s)$.
   1. Computations:

$$\begin{aligned}\zeta(s)^2&=\left(\sum_n\frac{1}{n^s}\right)\left(\sum_m\frac{1}{m^s}\right)\\&=\sum_{n,m}\frac{1}{(nm)^s}\\&=\sum_N\frac{|\{(n,m):nm=N\}|}{N^s}\\&=\sum_N\frac{d(n)}{N^s}\end{aligned}$$

   2. Computations:

$$\begin{aligned}\zeta(s)\zeta(s-1)&=\left(\sum_n\frac{1}{n^s}\right)\left(\sum_m\frac{1}{m^{s-1}}\right)\\&=\sum_{n,m}\frac{m}{(nm)^s}\\&=\sum_N\left(\sum_{m:\exists n\in\mathbb{N},nm=N}m\right)\frac{1}{N^s}\\&=\sum_N\frac{\sigma(n)}{N^s}\end{aligned}$$

Convex functions

This is a supplement to question 4.

A function $f:[a,b]\to\mathbb{R}$ is said to be convex if and only if for any $x,y\in[a,b]$ and $t\in[0,1]$

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)$$

A function $f:(a,b)\to\mathbb{R}$ is convex if and only if any of the following equivalent conditions hold for for $a\leq x<u<y\leq b$:

1. $$\det\begin{pmatrix}f(u)&u&1\\f(x)&x&1\\f(y)&y&1\end{pmatrix}\geq 0.$$
2. $$\frac{f(u)-f(x)}{u-x}\leq\frac{f(y)-f(x)}{y-x}$$
3. $$\frac{f(u)-f(x)}{u-x}\leq\frac{f(y)-f(u)}{y-u}$$

One just need to put $u=tx+(1-t)y$ and simplify, for example,

$$\begin{aligned}&\frac{f(u)-f(x)}{u-x}\leq\frac{f(y)-f(x)}{y-x}\\\iff&\frac{f(tx+(1-t)y)-f(x)}{(1-t)(y-x)}\leq\frac{f(y)-f(x)}{y-x}\\\iff&f(tx+(1-t)y)-f(x)\leq(1-t)(f(y)-f(x))\\\iff&f(tx+(1-t)\leq tf(x)+(1-t)f(y)\end{aligned}$$

For the first one,

$$\begin{aligned}&\det\begin{pmatrix}f(u)&u&1\\f(x)&x&1\\f(y)&y&1\end{pmatrix}\geq 0\\\iff&\det\begin{pmatrix}f(tx+(1-t)y)&tx+(1-t)y&1\\f(x)&x&1\\f(y)&y&1\end{pmatrix}\geq 0\\\iff&\det\begin{pmatrix}f(tx+(1-t)y)-tf(x)-(1-t)f(y)&0&0\\f(x)&x&1\\f(y)&y&1\end{pmatrix}\geq 0\end{aligned}$$

The result follows from expanding the determinant along the first row, and noticing that $x-y<0$